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Some Variational Formulas for Hausdorff Dimension, Topological Entropy, and SRB Entropy for Hyperbolic Dynamical Systems

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We derive several variational formulas for the topological entropy and SRB entropy of Axiom A flows on compact manifolds and for the Hausdorff dimension of basic sets for Axiom A diffeomorphisms on compact surfaces.

KEY WORDS: Topological entropy; SRB entropy; Hausdorff dimension; hyperbolic dynamical system; variational formulas.

INTRODUCTION

Topological and metric entropies and the Hausdorff dimension of invariant hyperbolic sets are among the most important global invariants of smooth dynamical systems. Topological entropy characterizes the total exponential complexity of the orbit structure with a single number. Metric entropy with respect to an invariant measure gives the exponential growth rate of the statistically significant orbits. The knowledge of entropies, especially in low dimensions, provides a wealth of quantitative structural information about the system. Such information includes the growth rate of periodic orbits, the existence of horseshoes, the growth rate of the volume of cells of various dimensions, ergodic components and factors with very stochastic behavior, etc.

In this note, we derive variational formulas for the topological entropy and SRB entropy of Axiom A flows on compact manifolds and for the Hausdorff dimension of basic sets for Axiom A diffeomorphisms on compact surfaces.

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All of these quantities are known to vary smoothly when the hyperbolic system is smoothly perturbed.[2–7] This is surprising, since these quantities are defined in global asymptotic terms, and *a priori*, one does not expect them to change smoothly. Hyperbolicity is essential, for there exist examples where entropies are discontinuous. We conjecture that *all* global asymptotic quantities associated with a hyperbolic dynamical system vary smoothly when the system is smoothly perturbed.

The variational formulas we derive are simple consequences of the methods of proof of smoothness of the above results using the thermodynamic formalism. For ease of exposition, we will state our results for Anosov flows, although the formulas apply to Axiom A flows with the obvious modifications.

Since the proofs of the smooth dependence of the entropies and Hausdorff dimension are based on the thermodynamic formalism, we begin by recalling some essential facts about pressure. We advise the reader to consult ref. 10 for a good introduction to the mathematical theory of pressure and to refs. 1 and 9 for a thorough exposition of the role of the thermodynamic formalism in hyperbolic dynamics.

Some Important Properties of Pressure[10]

Let ϕ be a homeomorphism of a compact metric space X.

(1) Let $C(X)$ denote the space of continuous functions on X. The quickest way to define the pressure of $f \in C(X)$ (with respect to the homeomorphism ϕ) is via the variational principle

$$P_\phi(f) = \sup_{\mu \in \mathcal{M}_\phi} \left[h_\mu(\phi) + \int f \, d\mu \right]$$

where $\mathcal{M}_\phi$ denotes the space of ϕ-invariant Borel probability measures and $h_\mu(\phi)$ denotes the measure-theoretic entropy of ϕ with respect to some measure μ. A measure μ for which this sup is attained is called an equilibrium measure for f.

(2) Pressure is an analytic function on Hölder continuous functions, i.e., $P\colon C^\alpha(X) \to \mathbb{R}$ defined by $f \to P_\phi(f)$ is a real analytic function. Hölder continuity is essential for this result. This result can be proved by showing that $\exp P_\phi(f)$ is the maximal isolated eigenvalue for the associated transfer operator for ϕ and then by applying standard results from perturbation theory.

(3) Let ϕ, ψ be homeomorphisms of compact metric spaces X, Y and let $\eta\colon X \to Y$ be a topological equivalence (homeomorphism) between ϕ and ψ, i.e., $\psi = \eta^{-1} \circ \phi \circ \eta$. Then for $g \in C(Y)$, $P_\psi(g) = P_\phi(g \circ \eta)$.[10]

(4) Let ϕ be an Axiom A diffeomorphism of a compact manifold M having a basic set Λ. For an Anosov diffeomorphism, $\Lambda = M$. Then for $f, g, h \in C^{\alpha}(\Lambda)$,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} P_{\phi}(f + \varepsilon g + \varepsilon^2 h) = \int_{\Lambda} g \, d\mu_f$$

where μ_f denotes the unique equilibrium measure for f.[9] This follows easily from our definition of pressure.

We now derive the three variational formulas.

1. TOPOLOGICAL ENTROPY FOR ANOSOV FLOWS

Let M be a compact manifold and let ϕ_λ^t, $-\varepsilon \leqslant \lambda \leqslant \varepsilon$, be a C^k perturbation of a C^k Anosov flow $\phi^t = \phi_0^t$. Let

$$h_T(\lambda) \stackrel{\text{def}}{=} h_T(\phi_\lambda^1)$$

denote the topological entropy of the time-1 map for the flow ϕ_λ^t. In ref. 4 the authors show that topological entropy varies (almost) as smoothly as the perturbation, i.e., the mapping $h_T: (-\varepsilon, \varepsilon) \to \mathbb{R}$ is C^{k-1}. In ref. 3 the authors derive several first derivative formulas for topological entropy. Below we derive another formula.

Let us briefly recall the smoothness argument in ref. 4. First, choose a Markov partition for the unperturbed flow ϕ^t and then use the conjugating homeomorphisms from structural stability to build Markov partitions for the perturbed flows such that all flows are realized as special flows over the same base (Σ_A, σ), where (Σ_A, σ) denotes a subshift of finite type. Denote by r_λ the return map between sections for ϕ_λ^t, or equivalently, the roof function for the symbolic flow corresponding to ϕ_λ^t. Our convention is that $r = r_0 =$ roof function for ϕ^t. The map r_λ is constructed using the maps from structural stability, which in ref. 4 were shown to depend smoothly on λ in the C^{α} topology.

It is again an exercise using our definition of pressure (variational principle) to show that for the shift map σ on Σ_A, $P_\sigma(-x r_\lambda) = 0 \Rightarrow x = h_T(\lambda)$.[4] Since pressure is an analytic function on C^{α} functions, the mapping $(-\varepsilon, \varepsilon) \times \mathbb{R} \to \mathbb{R}$ defined by $(\lambda, x) \to P_\sigma(-x r_\lambda)$ is smooth. One now applies the implicit function theorem to conclude that the map $h_T: (-\varepsilon, \varepsilon) \to \mathbb{R}$ is C^{k-1}. Mañé[4] has a clever argument to recover this last derivative and show that the mapping is C^k.

The following first derivative formula follows easily from this analysis:

Theorem 1. *First Derivative Formula for Topological Entropy.* Let M be a compact manifold and let ϕ_λ^t, $-\varepsilon \leqslant \lambda \leqslant \varepsilon$, be a C^2 perturbation of a C^2 Anosov flow ϕ^t. Then

$$\frac{d}{d\lambda}\bigg|_{\lambda=0} h_T(\lambda) = -h_T(0) \int_M \frac{dr_\lambda}{d\lambda}\bigg|_{\lambda=0} d\mu \bigg/ \int_M r\, d\mu$$

where μ denotes the $-h_T(0)\, r$ equilibrium measure for ϕ^t.

Proof. Since $P_\sigma(-h_T(\lambda)\, r_\lambda) \equiv 0$, it follows that

$$\frac{d}{d\lambda}\bigg|_{\lambda=0} P_\sigma(-h_T(\lambda)\, r_\lambda) = 0$$

Hence,

$$\begin{aligned} 0 &= \frac{d}{d\lambda}\bigg|_{\lambda=0} P_\sigma\bigg(-\bigg[h_T(0) + \lambda \frac{dh_T(\lambda)}{d\lambda}\bigg|_{\lambda=0} + O(\lambda^2)\bigg] \\ &\qquad \times \bigg[r + \lambda \frac{dr_\lambda}{d\lambda}\bigg|_{\lambda=0} + O(\lambda^2)\bigg]\bigg) \\ &= \frac{d}{d\lambda}\bigg|_{\lambda=0} P_\sigma\bigg(-h_T(0)\, r - \lambda\bigg[h_T(0) \frac{dr_\lambda}{d\lambda}\bigg|_{\lambda=0} + r \frac{dh_T(\lambda)}{d\lambda}\bigg|_{\lambda=0}\bigg] + O(\lambda^2)\bigg) \end{aligned}$$

It follows from Property 3 of pressure that

$$\int_M \bigg[h_T(0) \frac{dr_\lambda}{d\lambda}\bigg|_{\lambda=0} + r \frac{dh_T(\lambda)}{d\lambda}\bigg|_{\lambda=0}\bigg] d\mu = 0$$

The formula follows immediately. ∎

2. SRB ENTROPY FOR ANOSOV FLOWS

Let M be a compact manifold and let ϕ_λ^t, $-\varepsilon \leqslant \lambda \leqslant \varepsilon$, be a C^k perturbation of a C^k Anosov flow $\phi^t = \phi_0^t$. Let $\mu_\lambda^{\mathrm{SRB}}$ denote the SRB (Sinai–Ruelle–Bowen) measure associated to ϕ_λ^t (see ref. 1 for many characterizations of this important *physical* measure). If ϕ_λ^t preserves a family of smooth measures (as in the case of geodesic flows on negatively curved manifolds), then the smooth measures are the SRB measures. Let $h_{\mathrm{SRB}}(\lambda) \overset{\mathrm{def}}{=} h_{\mathrm{SRB}}(\phi_\lambda^1)$ denote the SRB entropy (measure-theoretic entropy with respect to the SRB measure) of the time-1 map of ϕ_λ^t. By the structural stability of Anosov flows, there exist C^α homeomorphisms $h_\lambda: M \to M$ and time changes $s_\lambda: \mathbb{R} \times M \to \mathbb{R}$ such that

$$\phi_\lambda^{s_\lambda(t,p)} \circ h_\lambda(p) = h_\lambda \circ \phi^t(p)$$

In refs. 5 and 6 the authors prove that for an important class of Anosov flows, the geodesic flows on negatively curved surfaces, the SRB entropy (Liouville entropy) varies as $C^{1+\varepsilon}$ (for all $\varepsilon > 0$) when the metric is smoothly perturbed. Extending the methods in ref. 4, Contreras,[2] using some ideas in ref. 7, showed that the SRB entropy changes (almost) as smoothly as the perturbation, i.e., the mapping $h_{\mathrm{SRB}}: (-\varepsilon, \varepsilon) \to \mathbb{R}$ is C^{k-1}. By slightly modifying his argument, one can obtain a first derivative formula for the SRB entropy.

Define functions

$$\chi_\lambda(x) = -\log \|D\phi_\lambda^1(x)\,|_{E_\lambda^u(x)}\|, \qquad \kappa_\lambda(x) = -\log \|D\phi_\lambda^{s_\lambda(1,*)}(x)\,|_{E_\lambda^u(x)}\|$$

where $E_\lambda^u(x)$ denotes the unstable distribution of ϕ_λ^t at x. The functions χ_λ and κ_λ have the same smoothness in x as the unstable distribution, which is typically only Hölder continuous. By imitating the nonsmooth phase space behavior with a perturbation, the same is true for the smoothness in λ. Mañé has shown the following amazing result:

Theorem.[7,2] The maps $(-\varepsilon, \varepsilon) \to C^\alpha(M)$ defined by $\lambda \to \chi_\lambda \circ h_\lambda$ and $\lambda \to \kappa_\lambda \circ h_\lambda$ are C^k.

Mañé proves this by showing that the bundle $E_\lambda^u \circ h_\lambda$ is the fixed point of the twisted graph transform

$$E \to D\phi_\lambda^{s_\lambda(1,*)} \circ E \circ \phi_\lambda^1$$

which he shows depends smoothly on λ.

The theme of Contreras' proof, which we slightly modify, is to *pull all the dynamics back to* ϕ^t. Let ν_λ be the ϕ^t equilibrium measure for $\kappa_\lambda \circ h_\lambda$, i.e.,

$$P_{\phi^1}(\kappa_\lambda \circ h_\lambda) = h_{\nu_\lambda}(\phi^1) + \int_M (\kappa_\lambda \circ h_\lambda)\, d\nu_\lambda$$

where $h_{\nu_\lambda}(\phi^t)$ denotes the measure-theoretic entropy of ϕ_λ^1 with respect to ν_λ.

It follows from properties 2 and 4 of pressure that

$$\left.\frac{d}{dt}\right|_{t=0} P_{\phi^1}(\kappa_\lambda \circ h_\lambda + t(\kappa_\lambda \circ h_\lambda)) = \int_M (\kappa_\lambda \circ h_\lambda)\, d\nu_\lambda$$

depends smoothly on λ. It also follows from property 3 of pressure and Pesin's entropy formula that

$$P_{\phi^1}(\kappa_\lambda \circ h_\lambda) = P_{\phi_\lambda^{s_\lambda(1,*)}}(\kappa_\lambda) = 0$$

Similarly,

$$h_{\nu_\lambda}(\phi^1) = h_{\mathrm{SRB}}(\phi_\lambda^{s_\lambda(1,*)})$$

This implies that

$$h_{\mathrm{SRB}}(\phi_\lambda^{s_\lambda(1,x)}) = -\int_M (\kappa_\lambda \circ h_\lambda)\, d\nu_\lambda$$

By Abramov's theorem, we have

$$h_{\mathrm{SRB}}(\phi_\lambda^{s_\lambda(1,*)}) = h_{\mathrm{SRB}}(\phi_\lambda^1) \int_M s_\lambda(1, x)\, d\mu_\lambda^{\mathrm{SRB}} = -\int_M (\kappa_\lambda \circ h_\lambda)\, d\nu_\lambda$$

The following formula follows by differentiating the above expression:

Theorem 2. *First Derivative Formula for SRB Entropy.* Let M be a compact manifold and let ϕ_λ^t, $-\varepsilon \leqslant \lambda \leqslant \varepsilon$, be a C^2 perturbation of a C^2 Anosov flow ϕ^t. Let $\mu_\lambda^{\mathrm{SRB}}$ denote the SRB measure associated to ϕ_λ^t, and let ν_λ be the ϕ^t equilibrium measure for $\kappa_\lambda \circ h_\lambda$. Then

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} h_{\mathrm{SRB}}(\lambda) = -h_{\mathrm{SRB}}(0) \left.\frac{d}{d\lambda}\right|_{\lambda=0} \int_M s_\lambda(1, p)\, d\mu_\lambda^{\mathrm{SRB}} - \left.\frac{d}{d\lambda}\right|_{\lambda=0} \int_M (\kappa_\lambda \circ h_\lambda)\, d\nu_\lambda$$

3. HAUSDORFF DIMENSION OF BASIC SETS

Let f be an Axiom A *surface* diffeomorphism having a basic set Λ. Define functions

$$\chi^u(x) = -\log \|Df(x)|_{E^u(x)}\|, \qquad \chi^s(x) = \log \|Df(x)|_{E^s(x)}\| \qquad \text{on } \Lambda$$

where $E^u(x)$ and $E^s(x)$ denote the unstable and stable distributions through x. McCluskey and Manning[8] gave an interpretation of the Hausdorff dimension $HD(\Lambda)$ in terms of the pressure of f, $\chi^u(x)$, and $\chi^s(x)$. They showed that $HD(\Lambda) = \delta^u + \delta^s$, where δ^u and δ^s are the unique numbers such that $P_f(\delta^u\chi^u)=0$ and $P_f(\delta^s\chi^s)=0$. They also showed that δ^u and δ^s, and hence $HD(\Lambda)$, vary continuously when f is smoothly perturbed.

Let f_λ, $-\varepsilon \leqslant \lambda \leqslant \varepsilon$, be a C^k perturbation of a C^k Axiom A diffeomorphism $f=f_0$ of a compact smooth surface M^2, and let Λ denote a basic set for f. By structural stability of hyperbolic sets, there exist basic sets Λ_λ for the maps f_λ, and homeomorphisms $h_\lambda\colon \Lambda \to \Lambda_\lambda$ such that on Λ, $h_\lambda \circ f = f_\lambda \circ h_\lambda$. Let $HD(\Lambda_\lambda)=\delta^u_\lambda+\delta^s_\lambda$ denote the Hausdorff dimension of the basic set Λ_λ. Mañé[7] showed that the map $\lambda \to HD(\Lambda_\lambda)$ is C^k.

We recall the heart of Mañé's argument.[7] It follows from definitions and property 3 of pressure that

$$0=P_{f_\lambda|\Lambda_\lambda}(\delta^u_\lambda\chi^u_\lambda)=P_{f|\Lambda}(\delta^u_\lambda(\chi^u_\lambda\circ h_\lambda))$$

Recalling that pressure is an analytic function on Hölder continuous functions, and recalling the theorem of Mañé in the preceding section, it follows that the mapping $(-\varepsilon,\varepsilon)\times\mathbb{R}\to\mathbb{R}$ defined by $(\lambda, x)\to P(x(\chi^u_\lambda\circ h_\lambda))$ is smooth. One now applies the implicit function theorem to conclude that δ^u_λ and δ^s_λ depend smoothly on λ.

The following variational formula follows immediately from his method of proof:

Theorem 3. *First Derivative Formula for Hausdorff Dimension.* We have

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0}\delta^u_\lambda=-\delta^u\int_\Lambda\left.\frac{d(\chi^u_\lambda\circ h_\lambda)}{d\lambda}\right|_{\lambda=0}dv^u\Big/\int_\Lambda\chi^u\,dv^u$$

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0}\delta^s_\lambda=-\delta^s\int_\Lambda\left.\frac{d(\chi^s_\lambda\circ h_\lambda)}{d\lambda}\right|_{\lambda=0}dv^s\Big/\int_\Lambda\chi^s\,dv^s$$

and hence

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0}HD(\Lambda_\lambda)=\left.\frac{d}{d\lambda}\right|_{\lambda=0}\delta^u_\lambda+\left.\frac{d}{d\lambda}\right|_{\lambda=0}\delta^s_\lambda$$

where v^u and v^s are the equilibrium measures for f associated to $\delta^u\chi^u$ and $\delta^s\chi^s$.

Proof of Theorem 3. The proof of Theorem 3 is also by direct calculation. Since $P(\delta^u_\lambda(\chi^u_\lambda\circ h_\lambda))\equiv 0$, it obviously follows that

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0}P(\delta^u_\lambda(\chi^u_\lambda\circ h_\lambda))=0$$

Hence,

$$0=\frac{d}{d\lambda}\bigg|_{\lambda=0} P\left(\left[\delta^u+\lambda\frac{d\delta^u_\lambda}{d\lambda}\bigg|_{\lambda=0}+O(\lambda^2)\right]\left[\chi^u+\lambda\frac{d(\chi^u_\lambda\circ h_\lambda)}{d\lambda}\bigg|_{\lambda=0}+O(\lambda^2)\right]\right)$$

$$=\frac{d}{d\lambda}\bigg|_{\lambda=0} P\left(\delta^u\chi^u+\lambda\left[\delta^u\frac{d(\chi^u_\lambda\circ h_\lambda)}{d\lambda}\bigg|_{\lambda=0}+\chi^u\frac{d\delta^u_\lambda}{d\lambda}\bigg|_{\lambda=0}\right]+O(\lambda^2)\right)$$

It follows from property 3 of pressure that

$$\int_\Lambda\left(\delta^u\frac{d(\chi^u_\lambda\circ h_\lambda)}{d\lambda}\bigg|_{\lambda=0}+\chi^u\frac{d\delta^u_\lambda}{d\lambda}\bigg|_{\lambda=0}\right)d\nu^u=0$$

Theorem 3 follows immediately. ▮

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